QUANTITATIVE NOISE ANALYSIS OF JITTER-INDUCED NONUNIFORMLY SAMPLED-AND-HELD SIGNALS

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ABSTRACT

Traditional jitter-noise analysis which mainly focuses on the clock jitter errors for impulse-sampled signals cannot accurately model the sampled-data systems that practically include nonuniform sample-and-hold effects. This paper presents a comprehensive and robust analysis of the imperfections of nonuniformly sampled-and-held signals due to clock jitter. Both the general signal to overall jitter noise ratio, and also the Spurious Free Dynamic Range subjected to narrow in-band noise tone will be derived in closed-forms. Finally, a practical analysis of the timing-skew effects in designing a 21.4 MHz IF sampled-data filter for radio applications will be addressed to illustrate the effectiveness of the derived formula.

1. INTRODUCTION

While the rapid evolution of electronic devices requires high-speed signal processing units, timing-jitter induced by inaccurate sampling clock creates a performance bottleneck especially for high-speed systems [1-5]. The jitter-induced nonuniformly sampling generates noise components in the sampled signal and thus degrades the signal quality.

Traditional analysis of sampling jitter effects deal with the nonuniformly impulse-sampled (IS) sequence [1-4], which is not practically suitable for general sampled-data systems with inherent output Sampled-and-Held (S/H) waveforms as shown in Fig.1 where $T=1/f_s$ is the nominal sampling period. Especially when the system output is not always further processed or sampled again by the next stage implying that the output holding at nonuniformly timing instants cannot be neglected. The signal frequency spectrum is also not simply a $\sin(x)/x$ shaped version of the spectrum of the corresponding impulse-sampled version, as in the case of traditional uniformly S/H signal [5].

Such kind of nonuniformly S/H signals can be typically observed in parallel sampled-data systems (e.g. in systems with front-end ADC and back-end DAC [6], and N-path sampled-data filters [8-10]) as shown in Fig. 2. Since the input and output sampling switches are usually driven by the same multi-phase generator, and thus the input and output sampling timing-errors are correlated, and the system output is in the nonuniformly sampled-and-held form. Such sampling timing errors in general have two different origins: One is imposed by purely random sampling jitter that results in an increased noise floor over all frequencies [3,4], while the other typically appears in parallel sampled-data systems, where the sampling time instants are unmatched but periodically fixed due to mismatches along different channels (resulting in the the periodic timing-skew fractional error sequence r_m, $m=0,1,...M$-1, as shown in Fig. 1, with period M which is usually equal to the parallel path number N) [6,7]. Such periodic timing-skew will cause noise tones that appear in frequency locations which are multiple of f_s/M, whereas for narrow band filtering application, one of the noise tone is normally in-band with the signal frequency, and not possible to be removed by the filter.
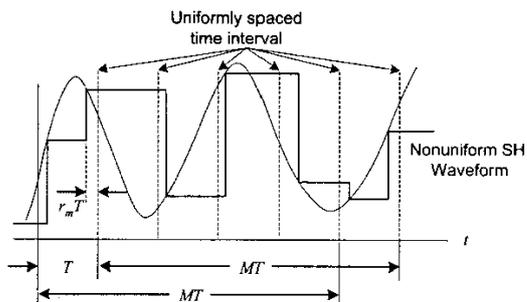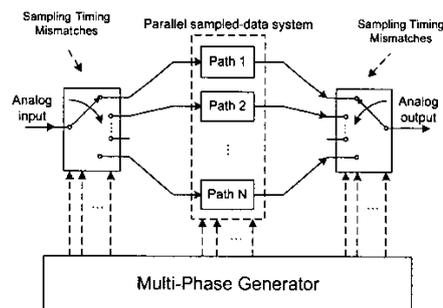

Fig.1 Nonuniformly Sampled-and-Held Signal

Fig.2 General parallel sampled-data systems structure

Thus, this paper will present a complete description of signal spectra of nonuniformly S/H waveform due to such sampling time errors. Both the spectra representation of such kind of signals will be given, and a closed-form expression of ratio of the signal to the noise induced by clock jitter will also be derived. Moreover, the closed-form expression of Spurious Free Dynamic Range (SFDR) subjected to in-band noise tone, which is suitable for the narrow-band filtering, will also be presented. Finally, a practical design example of a 21.4MHz IF sampled-data filter for radio communications will be analyzed to demonstrate the reliability of the proposed formulas.

2. JITTER NOISE ANALYSIS OF NONUNIFORMLY SAMPLED-AND-HELD SIGNAL

The spectra representation of nonuniformly S/H signals, as shown in Fig.1, were derived as [5]:

$$Y(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} A_k(\omega) \cdot X\left(\omega - k\frac{2\pi}{MT}\right) \quad (1)$$

where

$$A_k(\omega) = \frac{1}{M} \sum_{m=0}^{M-1} H_m(\omega) e^{-jkr_m\frac{2\pi}{M}} e^{-jkm\frac{2\pi}{M}} \quad (1a)$$

with

$$H_m(\omega) = \frac{2\sin\left(\omega(1 + r_{m+1} - r_m)T/2\right)}{\omega} e^{-j\omega(1+r_{m+1}-r_m)T/2} \quad (1b)$$

For a real input sinusoidal with frequency $\omega_0=2\pi f_o$, the noise tones are located at $\omega = \pm\omega_0+k(2\pi)/(MT)$, and to evaluate the SNR, all the noise components in the Nyquist band $[0, f_s/2]$ will be evaluated. Fig.3 shows a typical FFT spectra of a nonuniformly S/H sinusoid together with the corresponding impulse-sampled version with normalized frequency $a=f_o/f_s=0.2$, timing-skew period $M=8$, and standard deviation of $r_{rm}=0.1\%$.

The derivation of the expression of SNR of such kind of nonuniformly S/H waveform is rather complex and lengthy, so only the key steps of the procedure are presented here. Assume that $\omega_o\neq k(2\pi)/(MT)$, which means that the signal (and also the noise tones) is not *exactly* located at integer multiples of f_s/M. Substitute (1b) into (1a) with $\omega = \omega_o+k(2\pi)/(MT)$ and simplify it by Euler formula considering r_m and $e^{-jkm\frac{2\pi}{M}}$ is periodic with period $m = M$, which implies that

$$\sum_{m=0}^{M-1} e^{-jk(m+1)\frac{2\pi}{M}} = \sum_{m=1}^{M} e^{-jkm\frac{2\pi}{M}} = \sum_{m=0}^{M-1} e^{-jkm\frac{2\pi}{M}} \quad (2a)$$

$$\sum_{m=0}^{M-1} r_{m+1} e^{-jk(m+1)\frac{2\pi}{M}} = \sum_{m=1}^{M} r_m e^{-jkm\frac{2\pi}{M}} = \sum_{m=0}^{M-1} r_m e^{-jkm\frac{2\pi}{M}} \quad (2b)$$

and thus (1a) becomes

$$A_k(\omega_0 + k\frac{2\pi}{MT}) = \frac{1}{j(\omega_0 + k\frac{2\pi}{MT})M}\left\{\sum_{m=0}^{M-1} e^{-jkr_m\frac{2\pi}{M}} e^{-jkm\frac{2\pi}{M}}\right.$$
$$\left. -e^{-j\omega_0 T}\sum_{m=0}^{M-1} e^{-j\omega_0 r_m T} e^{-jkr_m\frac{2\pi}{M}} e^{j\omega_0 r_{m-1}T} e^{-jkm\frac{2\pi}{M}}\right\} \quad (3)$$

To find the SNR, both the expected value of signal power and also the sum of power of various noise components using (3) over the frequency range of $[0, f_s/2]$ need to be determined.

We can determine the expected value of signal power by assuming that r_m, $m = 0,1,2,...M-1$ to be M independent,
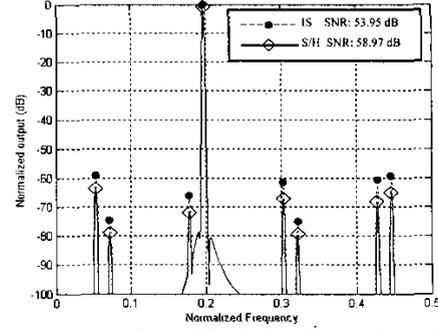

Fig.3 FFT spectra of nonuniformly impulse-sampled and sampled-and-held sinusoid (a=0.2, M=8, σ_{rm} = 0.1%)

identically distributed (*i.i.d*) random variables with Gaussian distribution of zero mean and standard deviation of σ_{rm} (=σ_t/T where σ_t is the standard deviation of timing jitter), with characteristic function given by $E[e^{j\omega_0 r_m T}] = e^{-\sigma_{rm}^2\omega_0^2 T^2/2}$. Substitute $k = 0$ into (3) and by evaluating the expected value of signal power $|A_0(\omega_0)|^2 = A_0(\omega_0)A_0^*(\omega_0)$ it yields:

$$E[|A_0(\omega_0)|^2] \approx \frac{4}{\omega_0^2}\sin^2(\omega_0 T/2) \quad (4)$$

assuming the timing-skew is small compared to T such that $\omega_0\sigma_{rm}T = 2\pi f_o\sigma_{rm}T\ll 1$. Using a similar approach, the expected value of the noise components can be expressed as follows:

$$E[|A_k(\omega_0 + k\frac{2\pi}{MT})|^2]$$

$$\approx \frac{2\sigma_{rm}^2 T^2}{M}\left\{2\sin^2(\omega_0 T/2) - 2\omega_0 T\sin^2(\omega_0 T/2)\cdot\frac{1-\cos(\omega_0 T + k\frac{2\pi}{M})}{\omega_0 T + k\frac{2\pi}{M}}\right.$$

$$\left. +\omega_0^2 T^2\cdot\frac{1-\cos(\omega_0 T + k\frac{2\pi}{M})}{\left(\omega_0 T + k\frac{2\pi}{M}\right)^2} - \omega_0 T\sin(\omega_0 T)\cdot\frac{\sin(\omega_0 T + k\frac{2\pi}{M})}{\omega_0 T + k\frac{2\pi}{M}}\right\} \quad (5)$$

and the SNR can be found by the following formula:

$$SNR = 10\log_{10}\left[\frac{|A_0(\omega_o)|^2}{\sum_k\left|A_k\left(\omega_o + k\frac{2\pi}{MT}\right)\right|^2 + \sum_k\left|A_k\left(-\omega_o + k\frac{2\pi}{MT}\right)\right|^2}\right] \quad (6)$$

To determine the expected values of total power of all noise components in the range of $[0, f_s/2]$, the sums in (5) are evaluated with value of k satisfying the following inequality:

$$0 < \pm\omega_0 T + k\frac{2\pi}{M} < \pi \text{ and } k \neq 0 \quad (7)$$

The direct calculation of the sum of noise power over this range is not realistic due to the complexity in (5). However, an important characteristic in (5) is that the expected value of noise components is a function of $\omega_0 T+k(2\pi)/M$, so for the case of purely random timing jitter ($M\rightarrow\infty$), the sum tends to an integral and it is possible to calculate the two sums in (6) by integrating (5) over the limit specified in (7). Thus, after the integration, the total noise power at $\omega=\omega_o+k(2\pi)/(MT)$ will be given by:
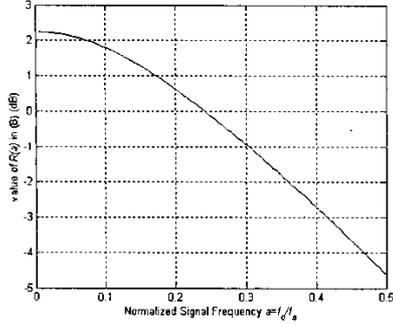
Fig.4 A plot of variation of the $R(a)$ in (9a)
vs. normalized frequency $a=f_o/f_s$

Fig.6 Comparison of simulated & calculated SNR vs. timing-
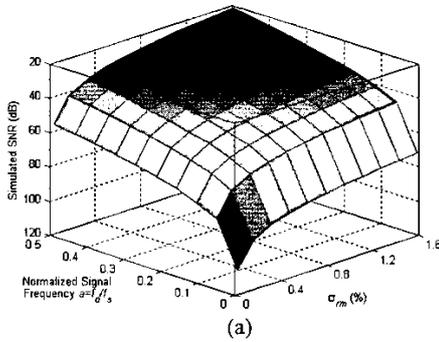skew period M and normalized frequency a ($\sigma_{rm} = 0.1\%$)

(a)

(b)

Fig. 5 (a) Simulated SNR & (b) Absolute error between the simulated and calculated SNR of nonuniformly sampled-and-held sinusoid
vs. normalized frequency a and standard deviation σ_{rm} (with 1000 times Monte-Carlo Simulations and $M=50$)

$$\sum_k \left| A_k \left(\omega_o + k\frac{2\pi}{MT} \right) \right|^2 = 4\sigma_{rm}^2 T^2 \sin^2(\pi a) \cdot [0.5 - 1.65a - \frac{1.85a}{\tan(\pi a)} + \frac{3.82a^2}{\sin^2(\pi a)}] \quad (8a)$$

and similarly the total noise power at $\omega = -\omega_o + k(2\pi)/(MT)$ is

$$\sum_k \left| A_k \left(-\omega_o + k\frac{2\pi}{MT} \right) \right|^2 = 4\sigma_{rm}^2 T^2 \sin^2(\pi a) \cdot [0.5 + 1.65a - \frac{1.85a}{\tan(\pi a)} + \frac{3.82a^2}{\sin^2(\pi a)}] \quad (8b)$$

By using (6), (4), (8a) and (8b), the closed-form expression of
SNR for jitter-induced nonuniformly S/H sinusoid is obtained by

$$SNR \approx 20\log_{10}\left(\frac{1}{2\pi a \sigma_{rm}} \right) + R(a) \quad (9)$$

where

$$R(a) = -10\log_{10}[1 - \frac{3.7a}{\tan(\pi a)} + \frac{7.64a^2}{\sin^2(\pi a)}] \quad (9a)$$

recalling that a is the normalized signal frequency f_o/f_s.

The derived SNR formula in (9) has two terms: the first
term is the main contribution of the overall SNR which depends
on both the normalized signal frequency and the standard
deviation of the timing-skew ratio (actually this part of SNR is
equal to the one of traditional nonuniformly impulse-sampled
signals [1]); the second term $R(a)$ is a factor that depends only on
signal frequency, and Fig. 4 shows a plot that gives an insight on
how this term contributes to the overall SNR. For $a \approx 0.25$, $R(a)$
≈ 0, which means that at this signal frequency the SNR for
nonuniformly S/H signal is equal to that for nonuniformly
impulse-sampled sequence. However, $R(a)$ which can increase or
in opposition degrade the SNR with maximum of 5 dB cannot be
neglected, as shown in Fig.4. Fig. 5 is the 3D plot showing both
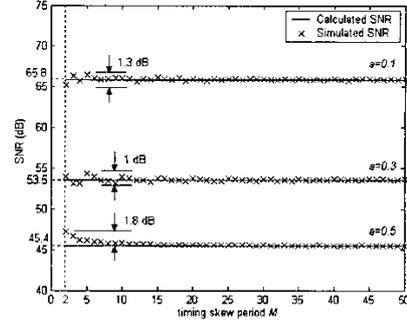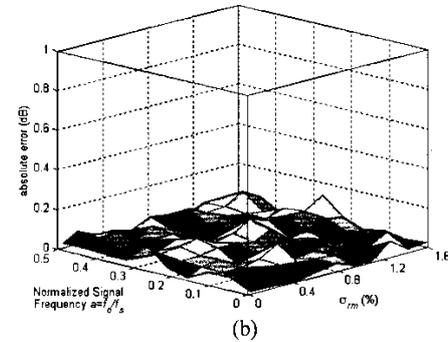the simulated SNR and the prediction error from the derived

formula in (9) vs. normalized signal frequency a and the standard
deviation σ_{rm} ($M=50$ to approximate the case of the purely
random jitter). The absolute error is well below 0.2 dB which
shows the effectiveness of the derived SNR formula.

Although the SNR derived in (9) with (9a) is for the case of
purely random jitter, it can be used also to approximate the SNR
due to periodic timing-skew. Fig.6 shows the MATLAB
simulation results of SNR as a function of timing-skew period M
and normalized signal frequency a together with the SNR
calculated from the derived formula ($\sigma_{rm}=0.1\%$). From this
figure, the simulation results converge to the calculated ones
when M is large, and even with low values of M, the formula can
approximate the SNR within 2 dB accuracy.

3. ANALYSIS OF IN-BAND NOISE TONE

One of the most common methods to construct a narrow-
band band-pass filter employs an N-path architecture [8-10] as
shown in Fig. 2 with pass-band center at the frequency of f_s/N.
However, one of the noise tones caused by periodic timing-skew
appears in-band with signal as shown in Fig. 3, thus influencing
the performance of N-path narrow-band filtering. To evaluate the
magnitude of this noise tone, suppose a signal is located at a
frequency of $\omega=\omega_o=2\pi f_o$. The value of k corresponding to the in-
band image can be calculated as follows (which is produced by
negative components of the signal):

$$\frac{1}{2}\frac{2\pi}{MT} < -\omega_0 + k\frac{2\pi}{MT} < \frac{3}{2}\frac{2\pi}{MT} \text{ or } 0.5 < -Ma + k < 1.5 \quad (10)$$

If the signal is placed close to the center frequency, then $a \approx$
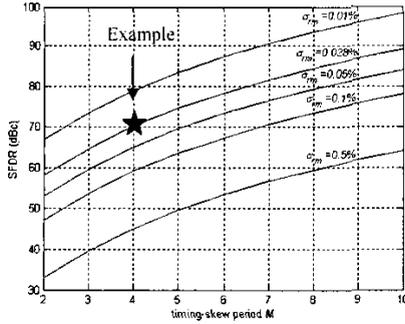$1/N=1/M$ and (10) will give $k = 2$.

Fig.7 A plot of variation of in-band SFDR
vs. timing-skew period M. and σ_{rm}

Fig.8 FFT spectrum of the 21.4 MHz 4-path FM radio IF filter
(signal frequency = 21.34 MHz, σ_{rm} = 0.028%)

The expected value of the noise tone can be found from (5) with $k = 2$ and $a \approx 1/M$ (using negative frequency components):

$$E[\left|A_2(-\omega_0 + 2 \cdot \frac{2\pi}{MT})\right|^2] = \frac{16\sigma_{rm}^2 T^2}{M}\sin^4\frac{\pi}{M} \qquad (11)$$

and the expected value of signal component results from (4) as:

$$E[\left|A_0(\omega_0)\right|^2] \approx \frac{M^2 T^2}{\pi^2}\sin^2\frac{\pi}{M} \qquad (12)$$

Thus, from (11) and (12) the Spurious Free Dynamic Range (SFDR) in the passband subjected to jitter is given by:

$$SFDR_{jitter} = 30\log_{10}M - 20\log_{10}\left(4\pi\sigma_{rm}\sin\frac{\pi}{M}\right)(dBc) \qquad (13)$$

Note that this equation is valid only when $a \approx 1/M$ and this is usually true for band-pass filter constructed with N-path techniques, as the signal is located near the frequency of $f_s/N = f_s/M$. Fig.7 shows a plot of (13) as function of the timing skew period M and standard deviation σ_{rm}, and from the figure a considerable reduction on the in-band noise tone caused by periodic timing-skew is possible via increasing the path number N (and the timing-skew period M).

4. APPLICATION TO AN FM RADIO IF SAMPLED-DATA FILTER

To illustrate the idea described in last section, consider a 21.4 MHz FM radio IF sampled-data filter with bandwidth of 200 kHz implemented using N-path techniques with $N = 4$, and overall sampling frequency f_s=85.6MHz. The timing skew among different paths will cause a noise tone to appear in the passband of the filter. In order to fulfill the requirements of portable application, the SFDR subjected to in-band noise tone should be greater than 70 dBc within band. Using (13) we can calculate the requirement of the σ_{rm} caused by periodic timing-skew that must be within 0.028%, or equivalently σ_t of 3.3 ps. This corresponds to the star mark shown in Fig. 7. Fig. 8 shows the simulation results corresponding to this example with the signal frequency of 21.34 MHz. The noise tone appears in-band at 21.46 MHz with a level of 70 dBc, which well matches the results calculated by the derived formula (13).

5. CONCLUSION

A complete and practical analysis of spectra of nonuniformly sampled-and-held signals due to timing jitter is presented in this paper. Because of the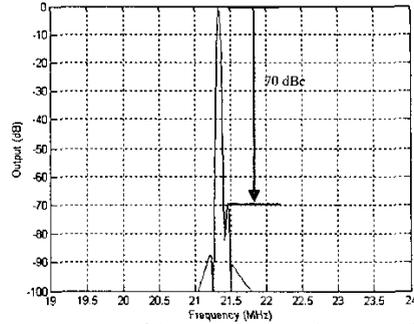 nature of the sampled-data signal and the jitter-errors in sampling time instants, the signal will be kept holding across nonuniform time-interval, and thus the output spectrum is not simply a $\sin(x)/x$ shaped version of the same signal in impulse-sampled version. The signal-to-noise-ratio of such kind of signals imposed by purely random jitter is derived in closed-form, and this formula can be also used to well approximate the SNR due to periodic timing-skew. The Spurious Free Dynamic Range (SFDR) subjected to in-band noise tone especially for narrow band N-path filtering is also derived in a closed-form, together with an example of designing a 21.4 MHz radio IF sampled-data filter to illustrate the effectiveness of the proposed formula. MATLAB simulation results show that the accuracy of the derived SNR expression is clearly under 0.2 dB, showing the effectiveness of the proposed formula.

6. REFERENCES

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